The Dirac oscillator. A relativistic version of the Jaynes-Cummings model

1999 J. Phys. A: Math. Gen. 32 5367

(http://iopscience.iop.org/0305-4470/32/28/314)

J. Phys. A: Math. Gen. **32** (1999) 5367–5382. Printed in the UK PII: S0305-4470(99)03010-3

The Dirac oscillator. A relativistic version of the Jaynes–Cummings model

P Rozmej† and R Arvieu‡

† Instytut Fizyki, Uniwersytet MCS, 20-031 Lublin, Poland
‡ Institut des Sciences Nucléaires, F-38026 Grenoble, France

E-mail: rozmej@tytan.umcs.lublin.pl and arvieu@isn.in2p3.fr

Received 24 March 1999

Abstract. The dynamics of wavepackets in a relativistic Dirac oscillator (DO) is compared with that of the Jaynes–Cummings model. The strong spin–orbit coupling of the DO produces the entanglement of the spin with the orbital motion similar to that observed in the model of quantum optics. The resulting collapses and revivals of the spin extend to a relativistic theory our previous findings on a nonrelativistic oscillator where they were known as *spin–orbit pendulum*. The Foldy–Wouthuysen transformation can be performed exactly for the DO. It produces the well known smoothing effect over the Compton wavelength. Thus, after this transformation, zitterbewegung disappears just as the components of the WP associated to negative energy states.

1. Introduction

This paper mixes together different popular models of quantum theory which have been developed separately and for different purposes and presents the Dirac oscillator as a relativistic version of the Jaynes–Cummings (JC) model [1] with, in addition to the regular properties of this model, some interesting new ones related to the relativistic description. The Dirac oscillator, (DO) first introduced by Ito *et al* [2], was later shown by Cook [3] to present unusual accidental degeneracies in its spectrum which were discussed from a supersymmetric viewpoint by Ui *et al* [4]. It was refreshed later by Moshinsky and Szczepaniak [5] and its symmetry Lie algebra was made explicit by Quesne and Moshinsky [6]. Moreno and Zentella [7] also showed that an exact Foldy–Wouthuysen (FW) transformation could be performed. More recently Nogami and Toyama [8] and Toyama *et al* [9] have studied the behaviour of wavepackets (WP) of the DO in the Dirac representation and in the FW representation in $1 + 1$ dimensions. The aim of these authors was to study WP which could possibly be coherent. This reduction of the dimension was brought about as an attempt to remove spin effects and to concentrate on the relativistic effects.

Our aim is to extend the work of [8, 9], to consider the full $3 + 1$ dimensions and show that an interesting new connection with the JC model can be made, as we explain shortly.

The degeneracies of the eigenvalues of the DO are due to a spin–orbit potential which is unusually large. In previous papers [10–12] we have analysed the time-dependent behaviour of WP in a nonrelativistic harmonic oscillator potential with a constant spin–orbit potential. We have shown that the behaviour of the spin shares a strong analogy with the observations made on the population of a two-level atom in a cavity where it can make a two-photon exchange [14].

For the relativistic DO the mechanism of collapses and revivals well known in the JC model is then expected to take place with some differences (see [15] for a general review of the JC model and for a full list of references). Since the energy spectrum is far from being linear the exact periodicity of the nonrelativistic oscillator is lost and the system evolves more on the lines of the regular JC model, i.e. without exact recurrences. In addition, the spin motion should also exhibit the famous zitterbewegung; this trembling motion should also be seen in the motion of the density of the wave as shown in [8, 9]. This effect should disappear in the FW representation.

The presence of components with negative energies in a relativistic WP solution of the Dirac equation with a potential leads to interesting new effects which cannot be interpreted satisfactorily using one-particle theory. In the scattering of a WP by a potential barrier step it is well known that there is a reflected current larger than the incident one and that there is a transmitted current for $V_0 > E + mc^2$ with a negative sign. This effect is well explained in the framework of hole theory (a full treatment of the Klein paradox is given in [17]). Studying the behaviour of a WP of the DO in a highly relativistic regime, we have found that the WP contains two pieces which circulate in the opposite sense and which are made with positive and negative energy states separately. When these components superpose, the magnitude of the WP is obviously larger than the part with positive energy states. The counter-rotating part is not present in the classical description, the same as the negative current which propagates under the barrier step.

2. Summary of results on the DO

In the following we will use the notations of [5, 16]. The time-dependent Dirac equation is written as

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = H_D\Psi = c[\boldsymbol{\alpha}\cdot(\boldsymbol{p} - \mathrm{i}m\omega\boldsymbol{r}\beta) + mc\beta]\Psi. \tag{1}$$

The components Ψ_1 and Ψ_2 of a spinor of energy E

$$\Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} \tag{2}$$

obey the equations

$$(E^2 - m^2c^4)\Psi_1 = \left[c^2(\boldsymbol{p}^2 + m^2\omega^2\boldsymbol{r}^2) - 3\hbar\omega mc^2 - \frac{4mc^2\omega}{\hbar}(\boldsymbol{L}\cdot\boldsymbol{S})\right]\Psi_1 \tag{3a}$$

$$(E^2 - m^2c^4)\Psi_2 = \left[c^2(\boldsymbol{p}^2 + m^2\omega^2\boldsymbol{r}^2) + 3\hbar\omega mc^2 + \frac{4mc^2\omega}{\hbar}(\boldsymbol{L}\cdot\boldsymbol{S})\right]\Psi_2. \tag{3b}$$

These components are thus the eigenstates of a spherical HO with a spin–orbit coupling term, respectively $\pm 2\omega/\hbar$. These large coupling strengths are responsible for the unusual degeneracies of the levels. The spectrum depends on a single parameter r defined as

$$r = \frac{\hbar\omega}{mc^2}. \tag{4}$$

Spectrum and degeneracies and the building of Ψ_1 are well described in [5, 6]. For an eigenstate of energy E_{nlj}, Ψ_1 can also be labelled by n, the usual total number of quantas of the 3D oscillator, by the orbital and total angular momentum l and j and by the component m of j_z. In terms of r

$$E_{nlj} = mc^2\sqrt{rA + 1} \tag{5}$$

where A is defined as

$$A = 2(n - j) + 1 \qquad \text{if} \quad l = j - \tfrac{1}{2} \tag{6a}$$

$$A = 2(n + j) + 3 \qquad \text{if} \quad l = j + \tfrac{1}{2}. \tag{6b}$$

Thus the states which obey (6*a*) have an infinite degeneracy. Among them those with $n = l$ have the lowest value $E = mc^2$. The states which obey (6*b*) have, in contrast, a finite degeneracy.

For the eigenvalues of equation (3*b*) one should take care of the opposite sign of the spin–orbit potential and of the different sign of the constant term. The eigenvalue is written in terms of $n'l'$ and j' as

$$E_{n'l'j'} = mc^2\sqrt{rA' + 1} \tag{7}$$

with A' given by

$$A' = 2(n' - j') + 3 \qquad \text{if} \quad l' = j' - \tfrac{1}{2} \tag{8a}$$

$$A' = 2(n' + j') + 5 \qquad \text{if} \quad l' = j' + \tfrac{1}{2}. \tag{8b}$$

The need to make E and E' equal with different n and n', l and l' is fulfilled if one notices that Ψ_2 is connected to Ψ_1 by

$$|\Psi_2\rangle = -\mathrm{i}\frac{mc^2}{E + mc^2}\sqrt{2r}(\boldsymbol{\sigma} \cdot \boldsymbol{a})|\Psi_1\rangle. \tag{9}$$

The operator which couples Ψ_1 and Ψ_2 above is expressed as the scalar product of the spin operator $\boldsymbol{\sigma}$ with a vector annihilation operator $\boldsymbol{a}$ defined by

$$\boldsymbol{a} = \frac{1}{\sqrt{2}}\left[\frac{\boldsymbol{r}}{\sqrt{\hbar/m\omega}} + \mathrm{i}\frac{\boldsymbol{p}}{\sqrt{\hbar m\omega}}\right]. \tag{10}$$

In order to satisfy equations (3*a*) and (9) it is necessary that

$$j' = j \qquad n' = n - 1 \tag{11a}$$

$$l' = l + 1 \qquad \text{if} \quad l = j - \tfrac{1}{2} \tag{11b}$$

$$l' = l - 1 \qquad \text{if} \quad l = j + \tfrac{1}{2}. \tag{11c}$$

These conditions imply that for the ground state of the equation satisfied by Ψ_1 one has $\Psi_2 = 0$. This corresponds to the fact that the DO is a supersymmetric potential for which Ψ_1 and Ψ_2 have the same spectrum apart from the absence in the spectrum of Ψ_2 of the ground state eigenvalue of Ψ_1.

Beside equation (9) we have

$$|\Psi_1\rangle = \mathrm{i}\frac{mc^2}{E - mc^2}\sqrt{2r}(\boldsymbol{\sigma} \cdot \boldsymbol{a}\dagger)|\Psi_2\rangle. \tag{12}$$

Using $|\Psi_1\rangle = |nljm\rangle$ and inserting (9) into (12) one obtains

$$E^2 - m^2c^4 = (mc^2)^2 2r|\langle n'l'jm|(\boldsymbol{\sigma} \cdot \boldsymbol{a})|nljm\rangle|^2 \tag{13}$$

where $|n'l'jm\rangle$ is a normalized harmonic oscillator state which has its quantum numbers defined by (11). From (9) and (13) and using the phase conventions as in the reference book [16] one obtains

$$\langle n'l'jm|\Psi_2\rangle = \mathrm{sgn}\sqrt{\frac{E - mc^2}{E + mc^2}}. \tag{14}$$

The complex phase called sgn is defined by

$$sgn = -\mathrm{i} \qquad \text{if} \quad l' = l + 1 = j + \tfrac{1}{2} \tag{15a}$$

$$sgn = +\mathrm{i} \qquad \text{if} \quad l' = l - 1 = j - \tfrac{1}{2}. \tag{15b}$$

A normalized spinor with positive energy $E = +E_p$ can now be expressed as

$$\Psi_+(t) = \begin{bmatrix} \sqrt{\frac{E_p+mc^2}{2E_p}}|nljm\rangle \\ \mathrm{sgn}\sqrt{\frac{E_p-mc^2}{2E_p}}|n'l'jm\rangle \end{bmatrix} \exp\left(\frac{-\mathrm{i}E_p t}{\hbar}\right). \tag{16}$$

In a similar manner a spinor with negative energy $E = -E_p$ is written as

$$\Psi_-(t) = \begin{bmatrix} \sqrt{\frac{E_p-mc^2}{2E_p}}|nljm\rangle \\ -\,\mathrm{sgn}\sqrt{\frac{E_p+mc^2}{2E_p}}|n'l'jm\rangle \end{bmatrix} \exp\left(\frac{\mathrm{i}E_p t}{\hbar}\right). \tag{17}$$

It is interesting to note that the relative weights of the large and small components are formally expressed in terms of E_p exactly in the same manner as in the 1 + 1 model of [8, 9]. The energies are, however, given by equation (5) with conditions (6*a*) and (6*b*).

3. Study of a circular wavepacket. Theory

3.1. Definition of the WP

In the following we will study the time evolution of a circular WP of a special kind in the DO. For $t = 0$ we assume that the WP has a Gaussian shape with an average position $\boldsymbol{r}_0$ and an average momentum $\boldsymbol{p}_0$ such that the WP moves upon a circular trajectory if it is left free of spin and in a nonrelativistic HO. The WP is assumed to initially be an eigenstate of spin with an arbitrary direction defined by two complex numbers, α and β. Let the normalized WP be

$$\Psi(\boldsymbol{r}, 0) = \frac{1}{(2\pi)^{3/4}\sigma^{3/2}} \exp\left[-\frac{(\boldsymbol{r}-\boldsymbol{r}_0)^2}{2\sigma^2} + \mathrm{i}\frac{\boldsymbol{p}_0\cdot\boldsymbol{r}}{\hbar}\right] \begin{pmatrix}\alpha\\ \beta\\ 0\\ 0\end{pmatrix}. \tag{18}$$

It is simpler to choose the axis of coordinates such that

$$\boldsymbol{r}_0 = \boldsymbol{x}x_0. \tag{19}$$

The centroid x_0 is expressed in units of the natural width of the HO and of a parameter called N by

$$x_0 = \sqrt{N}\sigma = \sqrt{N}\sqrt{\hbar/(m\omega)} \tag{20}$$

while the average momentum is taken as

$$\boldsymbol{p}_0 = \boldsymbol{y}p_0 = \boldsymbol{y}\hbar\sqrt{N}/\sigma. \tag{21}$$

In (19) and (21) $\boldsymbol{x}$ and $\boldsymbol{y}$ denote the appropriate unit vectors.

The average angular momentum is then

$$\langle L_z\rangle = x_0 p_0 = N\hbar. \tag{22}$$

The partial wave expansion of the WP involves only waves for which $m = l$ and for which the total number of quantas of the oscillator is also l. It is given by

$$|\Psi(0)\rangle = \sum_{l=0}^{\infty} \lambda_l |n = l l m_l = l\rangle \begin{pmatrix}\alpha\\ \beta\\ 0\\ 0\end{pmatrix}. \tag{23}$$

The weights λ_l are given by

$$\lambda_l = (-1)^l \exp(-N/2)\frac{N^{l/2}}{\sqrt{l!}}. \tag{24}$$

In other words (18) is a coherent state of the HO. We have studied in [10–12] its time evolution assuming that the Hamiltonian is a nonrelativistic HO with a constant spin–orbit potential which depends upon a parameter κ as

$$V_{s.o.} = \kappa(\boldsymbol{L}\cdot\boldsymbol{\sigma}). \tag{25}$$

Associated with the DO there is a nonrelativistic Hamiltonian that we will define as the operator in the right side of (3a) divided by $2mc^2$ and which thus presents the same degeneracies as those given by equations (6a) and (6b).

3.2. Partial waves with spin up

Let us isolate in the WP (18) a partial wave l with spin up. A partial wave has two partners with different total angular momentum called $j_+ = l + \frac{1}{2}$ and $j_- = l - \frac{1}{2}$ with respective energies E_+ and E_-. E_+ is given for all the l present in (23) by

$$E_+ = mc^2 \qquad \text{in the relativistic theory} \tag{26a}$$

$$E_+ = 0 \qquad \text{in the nonrelativistic one.} \tag{26b}$$

A partial wave with spin up has $j = j_+$ and is an eigenstate of the Hamiltonian with energy E_+. It leads trivially to the phase $\exp(-\mathrm{i}E_+t) = \exp(-\mathrm{i}\omega_0 t)$ in case (26a) with $\omega_0 = mc^2/\hbar$.

3.3. Partial waves with spin down

This component is coupled to the two partners and we therefore need E_- which is given by

$$E_- = mc^2\sqrt{2r(2l+1)+1} \qquad \text{in the relativistic theory} \tag{27a}$$

$$E_- = mc^2 r(2l+1) = \hbar\omega(2l+1) \qquad \text{in the nonrelativistic one.} \tag{27b}$$

(i) Let us consider first a two-component spinor in the nonrelativistic theory:

$$\begin{pmatrix} 0 \\ |lll\rangle \end{pmatrix} = \frac{1}{\sqrt{2l+1}}|lj_+m_j = j_+ - \tfrac{1}{2}\rangle + \frac{\sqrt{(2l)}}{\sqrt{2l+1}}|lj_-m_j = j_-\rangle. \tag{28}$$

After a time t the spinor is given by

$$\begin{pmatrix} 0 \\ |lll\rangle \end{pmatrix}_t = \frac{1}{2l+1}\begin{pmatrix} \sqrt{2l}(\mathrm{e}^{-\mathrm{i}E_+t/\hbar} - \mathrm{e}^{-\mathrm{i}E_-t/\hbar})|llm_l = l-1\rangle \\ (\mathrm{e}^{-\mathrm{i}E_+t/\hbar} + 2l\mathrm{e}^{-\mathrm{i}E_-t/\hbar})|llm_l = l\rangle \end{pmatrix}. \tag{29}$$

One sees that the average components of $\boldsymbol{\sigma}$ are given by

$$\langle\sigma_x\rangle = 0 \qquad \langle\sigma_y\rangle = 0 \tag{30a}$$

$$\langle\sigma_z\rangle = -1 + \frac{16l}{(2l+1)^2}\sin^2[(2l+1)\omega t]. \tag{30b}$$

At times

$$t = \pi/[2\omega(2l+1)] + n\pi \tag{31}$$

the average of spin reaches its minimum and the spin has a maximum of entanglement with the orbital motion. The most important component in the spinor (29) is for large enough values of l the spinor

$$\begin{pmatrix} 0 \\ \frac{2l}{2l+1}\mathrm{e}^{-\mathrm{i}\omega(2l+1)t}|lll\rangle \end{pmatrix}. \tag{32}$$

The time-dependent part of the phase $\exp(-\mathrm{i}2l\omega t)$ in this spinor can be incorporated with the angular phase $\exp(\mathrm{i}l\phi)$ of the spherical harmonic. Therefore, the main effect for this partial wave with spin down is that it rotates around Oz with angular velocity 2ω. Thus we have shown that a partial wave with arbitrary α and β contains a part which is at rest and a second part which rotates around Oz with angular velocity 2ω. This is a particular case of our previous findings [12]: in the general case with arbitrary κ the two waves with opposite spin move around Oz, the spin-up part with angular velocity $\omega - \omega_{ls}$ and the spin-down one with $\omega + \omega_{ls}$. In the DO we have simply $\omega_{ls} = \omega$.

(ii) The corresponding relativistic spinor is written simply by adding two components which are initially zero:

$$\begin{pmatrix} 0 \\ |lll\rangle \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2l+1}} \begin{pmatrix} |lj_+j_+ - 1\rangle \\ 0 \\ 0 \end{pmatrix} + \sqrt{\frac{2l}{2l+1}} \begin{pmatrix} |lj_-j_-\rangle \\ 0 \\ 0 \end{pmatrix}. \tag{33}$$

The first spinor gets the phase $\exp(-\mathrm{i}\omega_0 t)$ while the second spinor requires an expansion in terms of spinors with positive and negative energy as analysed in [8, 9]. The coupling is expressed by using equations (16) and (17) in terms of a coefficient called a_l defined as

$$a_l = \frac{\sqrt{E_-^2 - m^2c^4}}{E_-} = \sqrt{\frac{2r(2l+1)}{1+2r(2l+1)}} = \sqrt{1 - \left(\frac{\omega_0}{\omega_l}\right)^2} \tag{34}$$

where we have defined ω_l by

$$\omega_l = \omega_0\sqrt{1+2r(2l+1)}. \tag{35}$$

The second term of (33) is written at time t as

$$\begin{pmatrix} |lj_-j_-\rangle \\ 0 \\ 0 \end{pmatrix}_t = \begin{pmatrix} (\cos\omega_l t - \mathrm{i}\frac{\omega_0}{\omega_l}\sin\omega_l t)|lj_-j_-\rangle \\ \operatorname{sgn} a_l \sin\omega_l t|l-1j_-j_-\rangle \end{pmatrix} \tag{36}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2l+1}}(\cos\omega_l t - \mathrm{i}\frac{\omega_0}{\omega_l}\sin\omega_l t)|lll-1\rangle \\ \sqrt{\frac{2l}{2l+1}}(\cos\omega_l t - \mathrm{i}\frac{\omega_0}{\omega_l}\sin\omega_l t)|lll\rangle \\ \operatorname{sgn} a_l \sin\omega_l t|l-1l-1l-1\rangle \\ 0 \end{pmatrix}. \tag{37}$$

Finally, reassembling the four parts of the spinor (33) one writes it as

$$\begin{pmatrix} 0 \\ |lll\rangle \\ 0 \\ 0 \end{pmatrix}_t = \frac{1}{2l+1} \begin{pmatrix} [\sqrt{2l}[\mathrm{e}^{-\mathrm{i}\omega_0 t} - (\cos\omega_l t - \mathrm{i}\frac{\omega_0}{\omega_l}\sin\omega_l t)]|lll-1\rangle \\ [\mathrm{e}^{-\mathrm{i}\omega_0 t} + 2l(\cos\omega_l t - \mathrm{i}\frac{\omega_0}{\omega_l}\sin\omega_l t)]|lll\rangle \\ \sqrt{2l(2l+1)}\operatorname{sgn} a_l \sin\omega_l t|l-1l-1l-1\rangle \\ 0 \end{pmatrix}. \tag{38}$$

Comparing (38) with (29) we see that time evolution creates a small component in the relativistic spinor which is initially zero while the effect in the large components is through the following replacement:

$$\mathrm{e}^{-\mathrm{i}E_-t/\hbar} \to \left(\cos\omega_l t - \mathrm{i}\frac{\omega_0}{\omega_l}\sin\omega_l t\right) \tag{39}$$

$$= \frac{1}{2}\mathrm{e}^{-\mathrm{i}\omega_l t}\left[1 + \frac{\omega_0}{\omega_l}\right] + \frac{1}{2}\mathrm{e}^{\mathrm{i}\omega_l t}\left[1 - \frac{\omega_0}{\omega_l}\right]. \tag{40}$$

With similar arguments to those which led us to equation (32) we obtain in the relativistic case two waves which rotate in the opposite sense with angular velocity 2ω. To reach this conclusion we must linearize ω_l by writing

$$\omega_l = \omega_0[1 + r(2l+1)] = \omega_0 + \omega(2l+1). \tag{41}$$

The part with positive frequency is now weighted by $\frac{1}{2}(1+\omega_0/\omega_l)$ and the new part which rotates in the opposite sense by $\frac{1}{2}(1-\omega_0/\omega_l)$. The third component of the spinor (38) also contains two waves with an opposite sense of rotation and amplitude $a_l/2$. Thus, a relativistic WP with $\alpha=\beta=1/\sqrt{2}$ dissociates into three parts instead of two as in the nonrelativistic evolution: there is a part with spin up which does not essentially move, while the part with spin down is divided into two waves moving in opposite directions.

3.4. Spin averages

We give below the expectation values of the operator $\boldsymbol{\sigma}$ for the circular WP (23) assuming $\alpha=\beta=1/\sqrt{2}$:

$$\langle\sigma_x\rangle=\sum_l|\lambda_l|^2\frac{1}{2l+1}\left[1+l\left(1+\frac{\omega_0}{\omega_l}\right)\cos(\omega_l-\omega_0)t+l\left(1-\frac{\omega_0}{\omega_l}\right)\cos(\omega_l+\omega_0)t\right] \qquad (42)$$

$$\langle\sigma_y\rangle=\sum_l|\lambda_l|^2\frac{l}{2l+1}\left[\left(1+\frac{\omega_0}{\omega_l}\right)\sin(\omega_l-\omega_0)t+\left(1-\frac{\omega_0}{\omega_l}\right)\sin(\omega_l+\omega_0)t\right] \qquad (43)$$

$$\begin{aligned}\langle\sigma_z\rangle=\sum_l|\lambda_l|^2\bigg\{&\frac{1}{2}+\frac{4l-1}{2(2l+1)^2}-\frac{\omega_0^2}{\omega_l^2}\frac{2l^2}{(2l+1)^2}\\&-\frac{2l}{(2l+1)^2}\bigg[\left(1+\frac{\omega_0}{\omega_l}\right)\cos(\omega_l-\omega_0)t+\left(1-\frac{\omega_0}{\omega_l}\right)\cos(\omega_l+\omega_0)t\\&+l\left(1-\frac{\omega_0^2}{\omega_l^2}\right)\cos 2\omega_l t\bigg]\bigg\}.\end{aligned} \qquad (44)$$

These formulae extend to the relativistic DO those already discussed in [10]. Because of the conservation of the total angular momentum there is no interference between the various l for the spin averages. Each partial wave depends on time because the energies of the spin–orbit partners are different. In the nonrelativistic case the time factors depend only on the differences $\omega_l-\omega_0=(2l+1)\omega$. Therefore, all these averages have period $2\pi/\omega$. After a time called the collapse time $\tau_c=\pi/(2\sqrt{2N})$ all the phases coming from all the partial waves are equally distributed and assuming high values of N all these averages are zero, i.e. there is a collapse of the spin! The average orbital angular momentum correspondingly obtains an increase in order to preserve the average total angular momentum. This exchange was called *the spin–orbit pendulum* since it occurs exactly periodically. In addition, we have also shown that for a time equal to π/ω the average spin is for high N opposite to its initial value with the same coherence time around this revival. This behaviour makes the spin–orbit pendulum analogous to the JC model with the frequencies $(2l+1)\omega$ playing the role of the Rabi frequencies.

For the DO we essentially obtain the same behaviour. However, the periodicity is totally broken for high and even for low values of r because of the terms involving $\omega_0+\omega_l$. This combination introduces in the spin motion high frequencies affecting each component with a different weight. The effect of this modulation is the well known zitterbewegung that can be seen in the DO on the observable of spin. This effect generally occurs as an interference term between positive and negative energy states. In our problem these energies are l dependent, a situation which is not present for a free particle. Therefore, it will involve many frequencies but the smallest of them is certainly $2\omega_0=2mc^2/\hbar$. Higher frequencies are also there as exhibited in figure 1. Note that (42) and (43) contain similar terms and similar weights. $\langle\sigma_x\rangle$ and $\langle\sigma_y\rangle$ will then present a similar time behaviour. The Oz component is, however, different since there is in (44) an extra term with frequency $2\omega_l$. This term will produce two effects: an extra high-frequency modulation and a partial revival at a time about half of the revival of

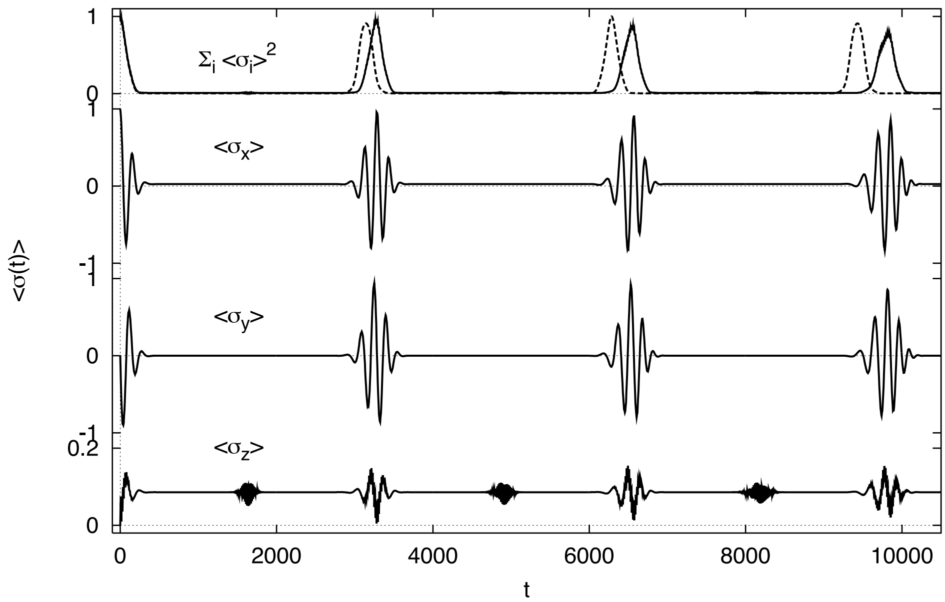

Figure 1. Time evolution of the average values of spin components for $N = 20$, $r = 0.001$ in the Dirac representation. Note that values of $\langle\sigma_z\rangle$ are five times enlarged. The nonrelativistic case (dashed curve) is included in the upper curve for comparison (in this case the period is $T = 2\pi/r = 6283.185$).

the spin. This new revival of a purely relativistic origin concerns only the z component which will oscillate rapidly while the other components are zero.

4. Circular WP. Numerical calculations

In order to exhibit the various effects discussed in the preceding section we have chosen the value $N = 20$ which provides a WP well concentrated in configuration space with an interesting spread of its partial waves. This value was also used thoroughly in our previous papers on the nonrelativistic oscillator [10–12]. For simplicity we use the units $\hbar = m = c = 1$. Therefore, our time units used in presented figures are proportional to $\omega^{-1} = r^{-1}$. For the nonrelativistic case the period of the motion is then $T = 2\pi/\omega = 2\pi/r$.

4.1. Spin averages in the Dirac representation

In figure 1 of [12] we have shown the 3D motion of the average spin vector and of the average orbital angular momentum vector for a nonrelativistic oscillator during an interval of time equal to $T_{ls}/2$ ($T_{ls} = 2\pi/\kappa$). The collapse of the spin is well exhibited in this figure by the fact that its average is zero during most of the time and its revival is seen in the opposite direction for $t = T_{ls}/2$. After twice that time the spin has regained its initial value and direction. During the interval where the spin collapses or revives it describes a curve in space. Qualitatively, the same features occur for the relativistic DO. However, the curve described by the spin vector is very sensitively dependent upon the use of relativistic energies in such a manner that the spin is not reversed at its first revival.

For the very low values of r, such as $r = 0.001$ in figure 1, we are very near the

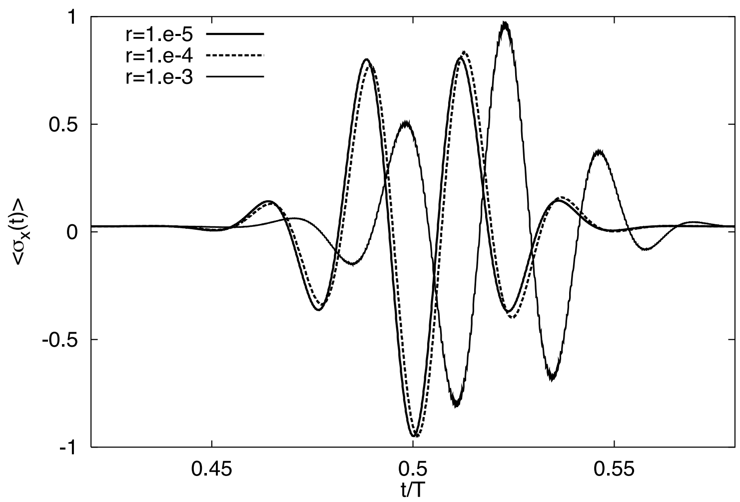

Figure 2. The details of the first revival around $t = T/2 = \pi/\omega$ for three small values of r.

nonrelativistic limit. One then observes a collapse of each component of the spin during an interval of time τ_c and a revival of the spin which has lost its periodicity because of the use of the relativistic energies. The first revival of $\langle\sigma_x\rangle$ at time $t = \pi/\omega$ indeed shows the spin with the same direction as the initial one. Figure 2 shows the details of the first revival for two additional values of r smaller than 0.001. For these smaller values, the spin gets its maximum in the direction opposite to the initial one while for $r = 0.001$, this direction has changed because of the simple transformation of a local maximum into an absolute maximum.

The relativistic effects produce a slow decrease in the amplitude of the revivals. There is already a quite sensible difference in the behaviour of $\langle\sigma_z\rangle$ with time. This component fluctuates much more rapidly because it is richer in frequencies than the other two. It also exhibits a small increase at a time about half of the recurrence time due to these higher components.

Each of these effects becomes more pronounced when the parameter r is given higher values. Figure 3 is for $r = 0.025$, figure 4 for $r = 0.5$. The components of the spin in the xOy plane oscillate much longer around each recurrence with a small period and the amplitudes of these recurrences decay. Again, the behaviour of $\langle\sigma_z\rangle$ is the most spectacular. One sees that on the average it does not exactly reach zero. The zitterbewegung is thus exhibited quite clearly in these time behaviours and the component $\langle\sigma_z\rangle$ qualitatively differs from the other two.

4.2. Probability densities

We have not attempted to detect the zitterbewegung in the change of the probability density with time. Indeed, since this effect involves high frequencies it is difficult to see in three dimensions. The reader is invited to read [8, 9] where it has been shown in 1 + 1 dimensions. The counter-rotating wave that was discussed in section 3.3 is however easily shown for high enough values of r. In figure 5 the total probability density of the WP at the particular average radius (19) is represented in spherical coordinates and for a few instants of time. What is shown was entirely explained in section 3.3 for each partial wave. A large part of the wave stays at the initial position, essentially the part with spin up. The wave is split into two waves which move in opposite directions and with the same angular velocity. They are centred around a circle with $\theta = \pi/2$. Two analyses of the WP have then been made and are represented in figures 6 and 7

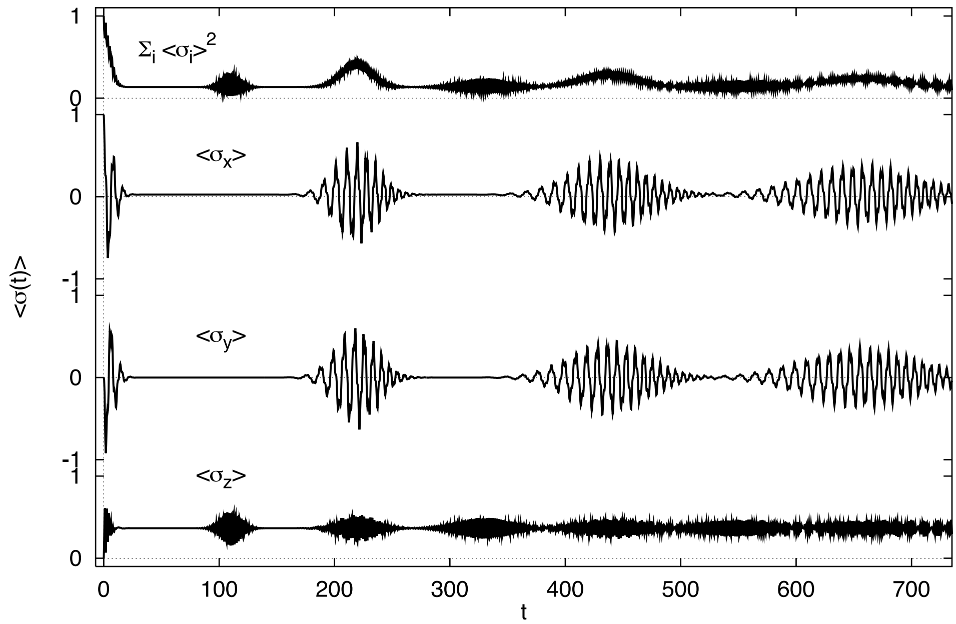

Figure 3. The same as in figure 1 but for $r = 0.025$.

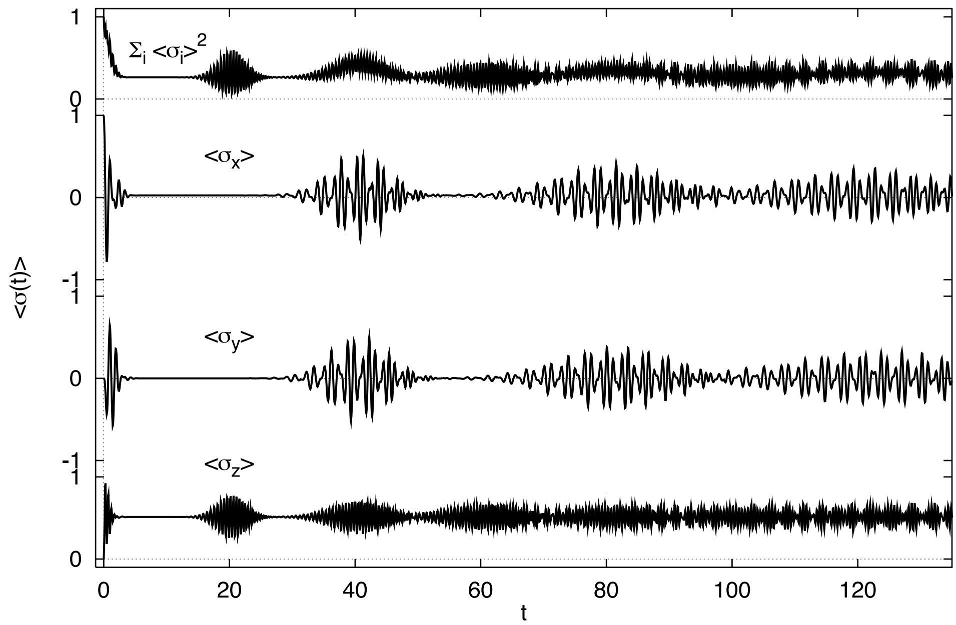

Figure 4. The same as in figure 1 but for $r = 0.5$.

for $t = 10$. There one finds that both of these moving parts are mainly localized in the second and third components of the spinor ($|c2\rangle$ and $|c3\rangle$, respectively) and that the counter-rotating part is almost entirely composed of negative energies. Here we are facing an effect totally

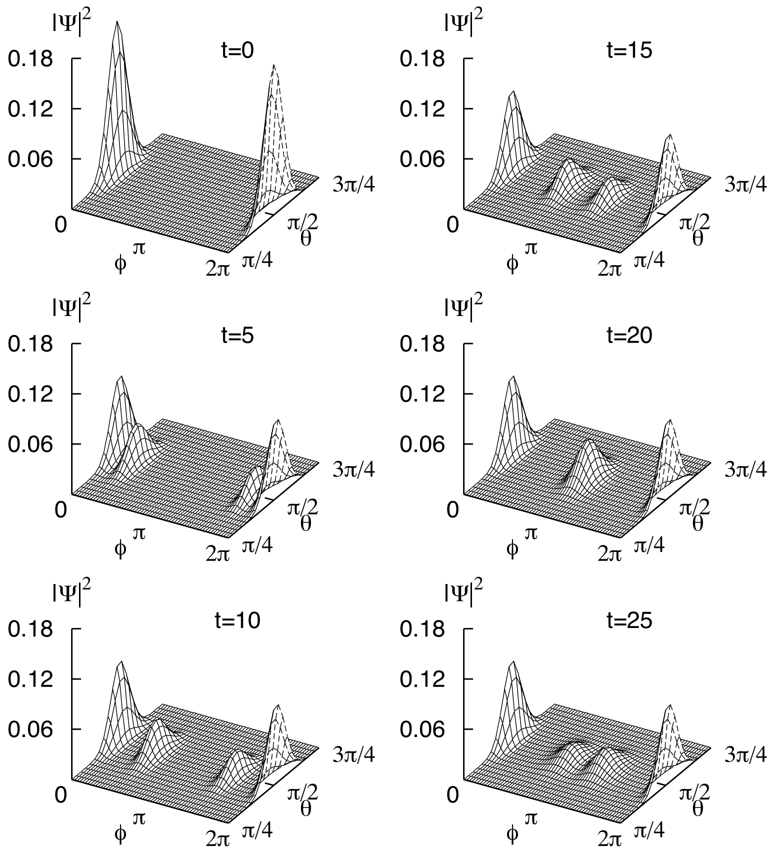

Figure 5. WP motion for $N = 20$, $r = 0.5$ in the Dirac representation. The total probability density $|\Psi|^2$ on the surface of the sphere with radius $r = x_0$ is shown. Note that motion of this *circular* WP remains close to the equator (narrow θ range).

absent from a nonrelativistic behaviour and not understood in one-particle theory. For lower values of r this part of the wave is hardly visible (not shown).

4.3. FW representation

The FW transformation [13] is well known to transform operators of the Dirac equation such as position, velocity, angular momentum and spin into new operators which permit an interpretation of the relativistic theory nearer to the classical one. The crucial point, well underlined in the original paper, is that the nonlocal character of this transformation produces a spread of the particle over the neighbourhood of dimension of the Compton wavelength. The velocity of the particle has convenient values and is not equal to the velocity of light. Moreover, components of negative energies are erased and the zitterbewegung disappears.

For a particle in an external electromagnetic field, the transformation requires an infinite sequence of transformations and the Hamiltonian becomes an infinite series of powers of $1/m$.

As derived in [7] a FW transformation can be performed exactly on the DO. The result obtained is very simple and makes calculations extremely easy. The small components Ψ_2 of the Dirac representation disappear and equation ($3a$) results as the only equation valid for Ψ_1. The spinor has still its eigenvalues given by (5) and the equations (28) and (29) should

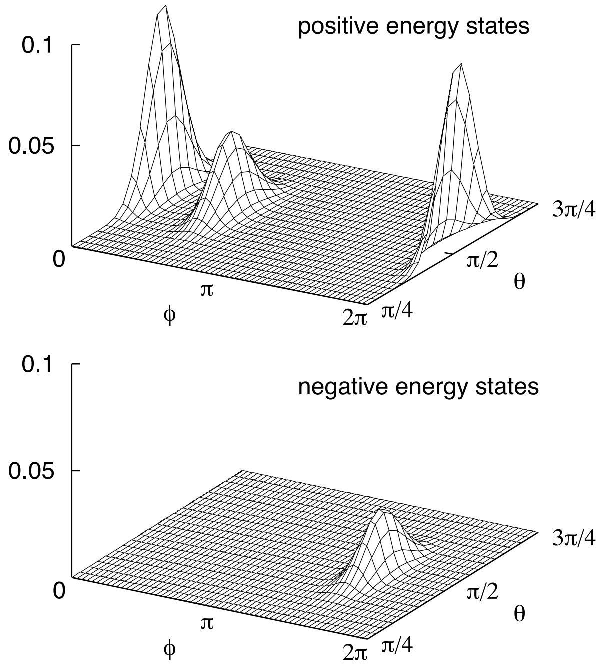

Figure 6. Contributions from positive and negative energy states for $N = 20$, $r = 0.5$ $t = 10$ in the Dirac representation.

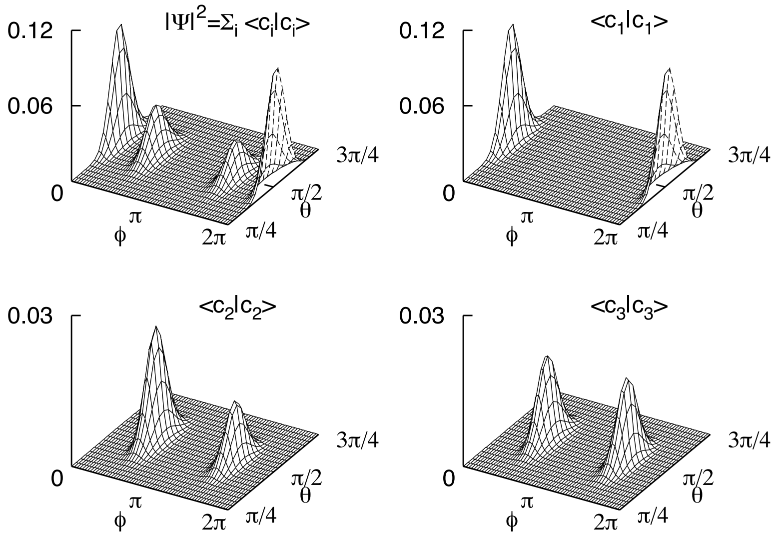

Figure 7. Contributions from all components of the bispinor Ψ (denoted as c_1, c_2, c_3 and c_4) for $N = 20$, $r = 0.5$ $t = 10$ in the Dirac representation. Note the different vertical scales and the fact that the contribution from the fourth component is zero.

be used for the spin averages. In other words, the only relativistic effects are the use of these energies. The effects introduced by negative energies disappear. This fact was already well discussed in the 1 + 1 dimension model [9]. A comparison of the calculation in the Dirac

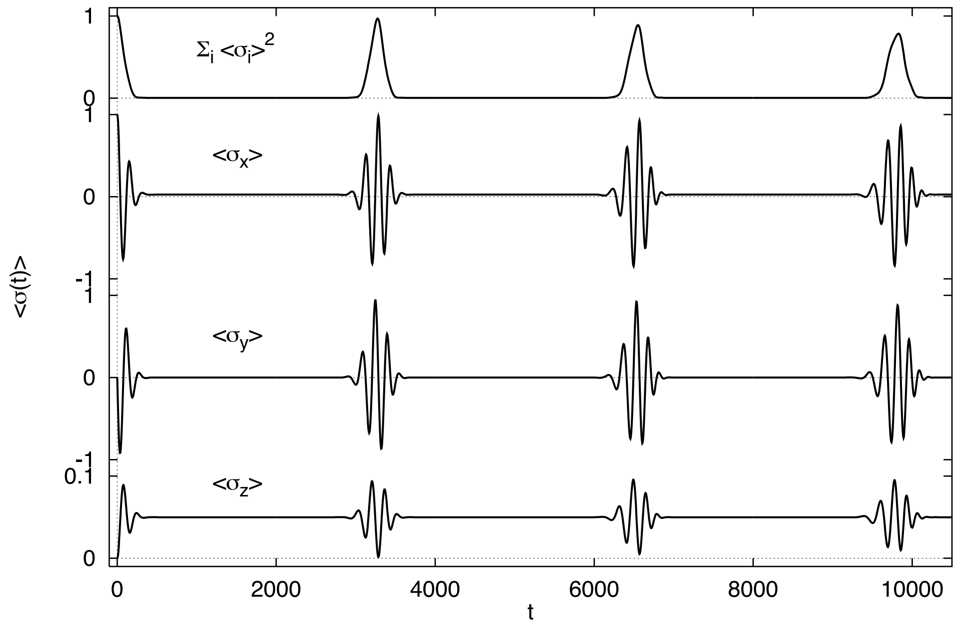

Figure 8. Time evolution of average values of spin components for $N = 20$, $r = 0.001$ in the FW representation. Note that values of $\langle\sigma_z\rangle$ are ten times enlarged.

and FW representation enables us to see the manifestations of the zitterbewegung exactly. Figures 8 and 9 present in FW representation the same cases as figures 1 and 4 in Dirac representation, respectively. Comparing the figures one sees that the rapid fluctuations have indeed been washed out. In the FW representation the behaviour of each component of the spin is now the same. Thus it is the use of the relativistic energies that now produce the rapid but regular oscillations of the spin as well as the spread and decay of its revival. It is natural in this context to expect the disappearance of the component rotating in the wrong sense. The WP are compared in figure 10 at time $t = 10$. Only the part which rotates in the positive sense is left in the FW representation.

4.4. Other spin directions

The formulae of sections 3.2 and 3.3 can be combined conveniently to provide the behaviour of a WP pointing initially in an arbitrary direction. Such a study does not lead to a new dynamics. One can in this way simply put more weight on the part with spin down which is the most variable part. For example, one can almost totally destroy the part not moving at the origin. Our choice of the initial direction has been made to see the components with spin up and down with the same magnitude. None of the other cases deserves a particular presentation.

5. Summary and conclusions

We have shown a new analogy between the relativistic DO and the JC model of quantum optics. The time evolution of the average spin associated to a WP in the DO is quite analogous to the time evolution of the occupation numbers of each of a two-level atom which interacts with an electromagnetic cavity. In the latter case the atom is entangled with the cavity while in the

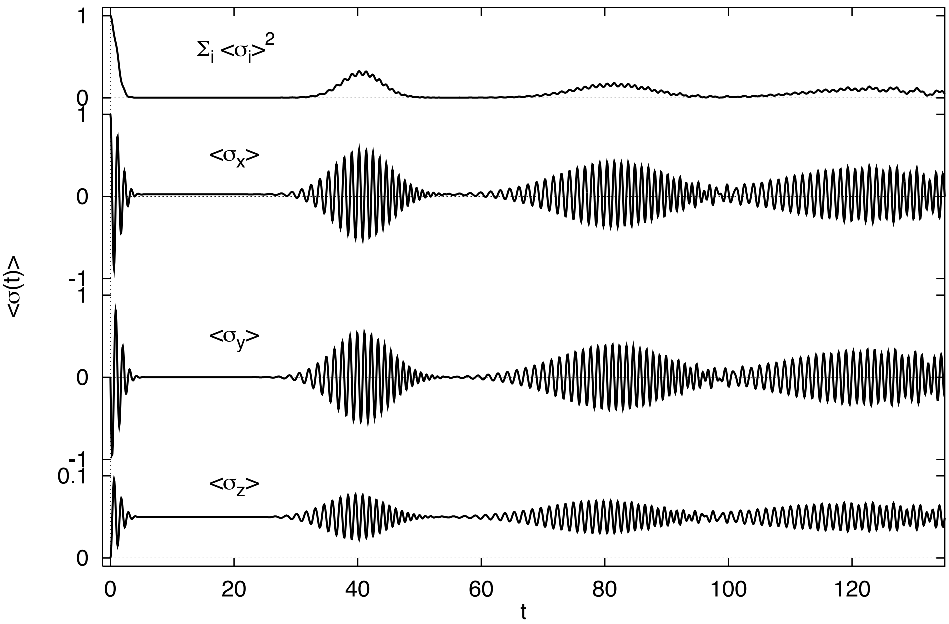

Figure 9. The same as in figure 8 but for $r = 0.5$.

former case the spin of the particle is entangled with its orbital angular momentum according to rules fixed by the Dirac equation. In the same way in both models the mechanism of collapses and revivals takes place. The collapse of the spin is compensated by a corresponding increase of the orbital average angular momentum. This balance occurs periodically in the nonrelativistic case [10–12]. We have proposed the name of spin–orbit pendulum for this effect. In the relativistic case the periodicity is destroyed. There is then a rich behaviour of the spin components which are submitted to zitterbewegung. For a WP initially thrown with its velocity in the xOy plane the z component of the spin contains more frequencies and therefore exhibits the most rapid oscillations.

Related to the relativistic description we have found the presence of a counter-rotating wave built mainly from negative energy states. This component is particularly large for the geometry of the WP we have used in our paper. Its presence is reminiscent of the transmitted propagating current observed under the barrier in the Klein paradox which is interpreted as a positron in hole theory [17]. To our knowledge it is the first time that such a nonclassical effect is observed in the case of a WP in a potential. We have been able to observe a similar (however weaker) effect for a WP in $1 + 1$ dimension. It is an open question whether this effect also exists for a WP in a Coulomb field. The smallness of the spin–orbit potential in this case may make observation difficult.

The FW transformation, performed exactly for the DO, produces its well known smoothing effect over the size of the Compton wavelength. The higher frequencies associated with zitterbewegung disappear and the counter-rotating component found in the Dirac representation is killed. Our conclusions totally confirm those of [8, 9]. The coherence of the WP is also lost because of the nonlinear relativistic energies. Therefore, coherent states of the harmonic oscillator generally spread. The counter-rotating wave also disappears completely with negative energies and the dynamics can be better interpreted with the ordinary one-particle interpretation. This dynamics then resembles the well known dynamics of the population

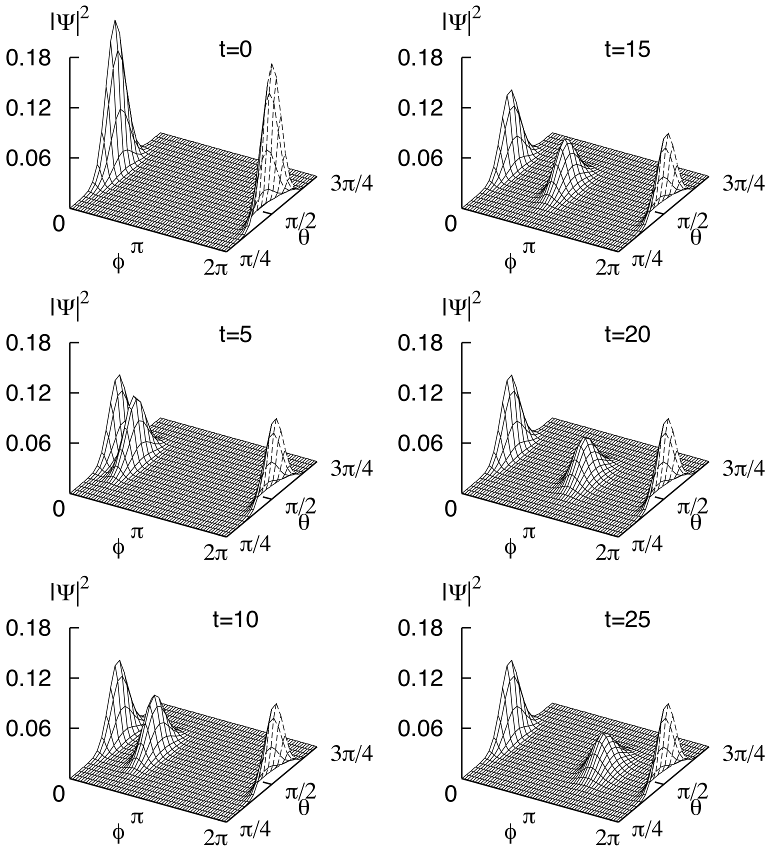

Figure 10. The same as in figure 5 but for the FW representation.

inversion of the JC model with Eberly revivals of Rabi oscillations [14]. There is an attempt by Toyama and Nogami to provide coherent relativistic WP of DO by using the inverse scattering method [18]. If these WP could also be defined for a 3 + 1 oscillator we would then probably have a relativistic spin–orbit pendulum with a dynamics similar to the nonrelativistic one. To our knowledge these WP have not yet been constructed in the case where we take all dimensions into account.

Acknowledgment

PR kindly acknowledges support of Polish Committee for Scientific Research (KBN) under the grant 2 P03B 143 14.

References

[1] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[2] Ito D, Mori K and Carriere E 1967 *Nuovo Cimento* A **51** 1119
[3] Cook P A 1971 *Lett. Nuovo Cimento* **1** 419
[4] Ui H and Takeda G 1984 *Prog. Theor. Phys.* **72** 266
[5] Moshinsky M and Szczepaniak A 1989 *J. Phys. A: Math. Gen.* **22** L817
[6] Quesne C and Moshinsky M 1990 *J. Phys. A: Math. Gen.* **23** 2263

[7] Moreno M and Zentella A 1989 *J. Phys. A: Math. Gen.* **22** L821
[8] Nogami Y and Toyama F M 1996 *Can. J. Phys.* **74** 114
[9] Toyama F M, Nogami Y and Coutinho F A B 1997 *J. Phys. A: Math. Gen.* **30** 2585
[10] Arvieu R and Rozmej P 1994 *Phys. Rev.* A **50** 4376
[11] Arvieu R and Rozmej P 1995 *Phys. Rev.* A **51** 104
[12] Rozmej P and Arvieu R 1996 *J. Phys. B: At. Mol. Opt. Phys.* **29** 1339
[13] Foldy L L and Wouthuysen S A 1950 *Phys. Rev.* **78** 29
[14] Knight P L 1986 *Phys. Scr.* T **12** 51
[15] Shore B W and Knight P L 1993 *J. Mod. Opt.* **40** 1195
[16] Moshinsky M and Smirnov Y F 1996 *The Harmonic Oscillator in Modern Physics* (New York: Harwood Academic)
[17] Greiner W 1990 *Relativistic Quantum Mechanics, Wave Equations* (Berlin: Springer)
[18] Nogami Y and Toyama FM 1999 *Phys. Rev.* A **59** 1056